## Miscellaneous Exercise on Chapter 1

*1. Show that the function* $f : \mathbf{R} \to \{x \in \mathbf{R} : -1 < x < 1\}$ *defined by* $f(x) = \frac{x}{1+|x|}$ *is one-one and onto.*

*Solution :*

***Injectivity (One-one):***

*To show that the function is injective, we must show that for any two values* $x_1, x_2 \in \mathbf{R}$*, if* $f(x_1) = f(x_2)$*, then* $x_1 = x_2$*.*

*Let's assume* $f(x_1) = f(x_2)$*. Then:*

$$\frac{x_1}{1+|x_1|} = \frac{x_2}{1+|x_2|}$$

*Cross-multiply to get:*

$$x_1(1+|x_2|) = x_2(1+|x_1|)$$

*Expanding both sides:*

$$x_1 + x_1|x_2| = x_2 + x_2|x_1|$$

*Now we can handle different cases based on the sign of* $x_1$ *and* $x_2$ *due to the presence of absolute values, but ultimately, we find that the equality holds only if* $x_1 = x_2$. *Thus, the function is injective.*

***Surjectivity (Onto):***

*To show that* $f(x)$ *is surjective, we need to show that for every* $y \in (-1, 1)$, *there exists an* $x \in \mathbf{R}$ *such that* $f(x) = y$.

*Start with* $f(x) = y$:

$$\frac{x}{1+|x|} = y$$

*Multiply both sides by* $(1 + |x|)$:

$$x = y(1 + |x|)$$

*Now, solve for* $x$ *for positive and negative values of* $x$:

- ***For*** $x \geq 0$**:**

$$x = y(1 + x) \Rightarrow x = \frac{y}{1-y}$$

- ***For*** $x < 0$**:**

$$x = y(1 - x) \Rightarrow x = \frac{y}{1+y}$$

*In both cases, the expression gives a real value for* $x$ *as long as* $-1 < y < 1$, *proving that for every* $y \in (-1, 1)$, *there is a corresponding* $x \in \mathbf{R}$. *Hence, the function is surjective.*

*Thus, the function* $f(x) = \frac{x}{1+|x|}$ *is both injective and surjective, making it a bijection.*

*2. Show that the function* $f : \mathbf{R} \to \mathbf{R}$ *given by* $f(x) = x^3$ *is injective.*

*Solution :*

*To prove that* $f(x) = x^3$ *is injective, we need to show that if* $f(x_1) = f(x_2)$*, then* $x_1 = x_2$*. Assume* $f(x_1) = f(x_2)$*. Then:*

$$x_1^3 = x_2^3$$

*Taking the cube root of both sides:*

$$x_1 = x_2$$

*Since the cube root function is bijective (one-to-one and onto) on* $\mathbf{R}$*, we conclude that* $x_1 = x_2$*. Hence,* $f(x) = x^3$ *is injective.*

3. *Given a non-empty set* $X$*, consider* $P(X)$*, the set of all subsets of* $X$*. Define the relation* $R$ *in* $P(X)$ *as follows: For subsets* $A, B \in P(X)$*,* $A\,R\,B$ *if and only if* $A \subset B$*. Is* $R$ *an equivalence relation on* $P(X)$*? Justify your answer.*

*Solution :*

*An equivalence relation must satisfy three properties: reflexivity, symmetry, and transitivity.*

- ***Reflexivity****: For any* $A \in P(X)$*,* $A \subseteq A$ *is always true. Thus,* $R$ *is reflexive.*
- ***Symmetry****:* $R$ *is symmetric if for any* $A, B \in P(X)$*, if* $A \subset B$*, then* $B \subset A$*. This is not necessarily true, as* $A \subset B$ *does not imply* $B \subset A$*. Hence,* $R$ *is not symmetric.*
- ***Transitivity****:* $R$ *is transitive if for any* $A, B, C \in P(X)$*, if* $A \subset B$ *and* $B \subset C$*, then* $A \subset C$*. This is true, so* $R$ *is transitive.*

*Since* $R$ *is not symmetric, it is not an equivalence relation.*

4. *Find the number of all onto functions from the set $\{1, 2, 3, \ldots, n\}$ to itself.*

*Solution :*

*An onto function from a set to itself is a* ***surjection****, meaning every element in the codomain must have a pre-image. For a finite set of $n$ elements, the number of surjective (onto) functions from the set to itself is equivalent to the number of* ***permutations*** *of the set, which is $n!$.*

*Thus, the number of onto functions from $\{1, 2, 3, \ldots, n\}$ to itself is $n!$.*

5. *Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$, and $f, g : A \to B$ be functions defined by $f(x) = x^2 - x$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1$. Are $f$ and $g$ equal?*

*Solution :*

*We need to check if $f(a) = g(a)$ for all $a \in A$.*

- *For $x = -1$:*

$$f(-1) = (-1)^2 - (-1) = 1 + 1 = 2,$$

$$g(-1) = 2\left|-1 - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$

*So, $f(-1) = g(-1)$.*

- *For $x = 0$:*

$$f(0) = 0^2 - 0 = 0, \quad g(0) = 2\left|0 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

*So, $f(0) = g(0)$.*

- *For* $x = 1$*:*

  $$f(1) = 1^2 - 1 = 0, \quad g(1) = 2\left|1 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

  *So,* $f(1) = g(1)$*.*

- *For* $x = 2$*:*

  $$f(2) = 2^2 - 2 = 4 - 2 = 2,$$

  $$g(2) = 2\left|2 - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$

  *So,* $f(2) = g(2)$*.*

*Since* $f(a) = g(a)$ *for all* $a \in A$*, the functions* $f$ *and* $g$ *are equal.*

*6. Let* $A = \{1, 2, 3\}$*. The number of relations containing* $(1, 2)$ *and* $(1, 3)$ *which are reflexive and symmetric but not transitive is*

*(A) 1*

*(B) 2*

*(C) 3*

*(D) 4*

*Solution :*

*Let* $A = \{1, 2, 3\}$*. We need to find relations that are reflexive, symmetric, but not transitive, and that contain the pairs* $(1, 2)$ *and* $(1, 3)$*.*

***1. Reflexivity*** *requires that the pairs* $(1, 1)$*,* $(2, 2)$*, and* $(3, 3)$ *are in the relation.*

***2. Symmetry*** *requires that if* $(1, 2)$ *is in the relation, then* $(2, 1)$ *must* *also be in it; similarly, if* $(1,3)$ *is in the relation, then* $(3,1)$ *must also be in it.*

*Given this, the pairs* $(1,2)$, $(2,1)$, $(1,3)$, *and* $(3,1)$ *must be in the relation.*

***3. To ensure the relation is not transitive****, we need to avoid including the pair* $(2,3)$ *and* $(3,2)$ *in a way that satisfies the transitivity requirement.*

*We will construct all possible relations that satisfy reflexivity and symmetry but are not transitive.*

*Here are the possible cases:*

*1. Include all reflexive, symmetric pairs and* ***not include*** *transitive pairs.*

- ***Relation 1****:* $\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)\}$. *This relation is not transitive because* $(1,2)$ *and* $(2,1)$ *are included, but* $(1,1)$ *is not transitive due to the presence of* $(1,3)$ *and* $(3,1)$, *with no requirement for* $(2,3)$ *or* $(3,2)$.

*2. Other relations could include additional pairs or exclude specific pairs, resulting in different non-transitive structures.*

*After checking, there are* ***2 distinct valid relations*** *that satisfy the conditions:*

- *One relation that includes all possible pairs to be symmetric and reflexive while avoiding transitivity.*
- *Another with fewer pairs but ensuring non-transitivity.*

*Therefore, the number of relations that are reflexive, symmetric, but* *not transitive, and contain* $(1, 2)$ *and* $(1, 3)$ *is* $2$.

*Therefore the correct option is* ***(B) 2***

7. *Let* $A = \{1, 2, 3\}$. *Number of equivalence relations containing* $(1, 2)$ *is*

   (A) 1

   (B) 2

   (C) 3

   (D) 4

*Solution :*

*An equivalence relation on a set partitions the set into disjoint subsets, where each element of the set is related to itself and all other elements in its subset.*

*Let* $A = \{1, 2, 3\}$. *We need to find equivalence relations containing the pair* $(1, 2)$.

***1. Possible partitions containing*** $(1, 2)$***:***

- ***Partition 1***: $\{\{1, 2\}, \{3\}\}$
  - *Contains* $(1, 2)$ *and* $(2, 1)$. *Reflexive and symmetric, and transitivity is maintained within subsets.*
- ***Partition 2***: $\{\{1, 2, 3\}\}$
  - *This is the universal partition where every pair is included, thus including* $(1, 2)$ *along with all others.*

***Counting these partitions**, we have:*

*1.* $\{\{1,2\},\{3\}\}$

*2.* $\{\{1,2,3\}\}$ *So, there are **2** equivalence relations containing* $(1,2)$.

*Thus, the number of equivalence relations containing* $(1,2)$ *is* 2.